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# Phase space transport and control of escape from a potential well

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**Abstract**

A framework for controlling a nonlinear dynamical system against escape from a potential well is presented based on reducing the phase space transport across the separatrix associated with the potential well. A bandlimited open-loop control with finite lag is considered for systems with weak additive stationary forcing including, specifically, the colored Gaussian case. The related multiplicative, closed-loop control problem is shown to reduce to an open-loop problem. A numerical example based on the Duffing oscillator is presented to illustrate the theory.

*Keywords:* Control; Stability; Chaos; Escape

## 1. Introduction

Two broad approaches exist for controlling chaos and, hence, modifying system stability in nonlinear dynamical systems. The first, which attempts to select and stabilize unstable orbits or steady states, is described in [14] and references therein. The second seeks to control system stability by modifying the system Melnikov process, typically through forced resonant parametric oscillation [3,9]. The existence of zeros of the Melnikov process is a renowned computable criterion for the presence of chaos. The second approach specifically aims to eliminate these zeros. Modification of the system Melnikov process to achieve stability has until now only been considered for sinusoidal forcing. This is in part due to a technical difficulty which arises with some other, more general classes of forcing. For ergodic Gaussian forcing, for example, the Melnikov process exhibits zeros at all forcing levels regardless of the control employed. The present work has two purposes. First, the difficulty encountered with the Melnikov process is removed by treating a functional – the flux factor – of the Melnikov process. This allows us to consider controls for external forcing belonging to the broad class of wide-sense stationary processes including, in particular, Gaussian forcing. This work's second purpose is to consider modifying the flux factor not for controlling chaos but rather for controlling against escape from a potential well. That this is possible is a remarkable result of the fact that the flux

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factor measures both the level of chaos and also, equivalently, the transport of phase space across the separatrix associated with a potential well.

For systems for which a potential energy can be defined, each well of the potential has in the absence of external forcing an associated region in phase space bounded by a separatrix. System states interior to the separatrix are identically those states within the corresponding potential well lacking sufficient energy for escape. States exterior to the separatrix correspond to states outside the well together with states in the well having too much energy to remain. The separatrix forms an impenetrable barrier between these two sets of states and if initially within the separatrix the system state never leaves. In the presence of weak external forcing this global structure persists, though now the separatrix is supplanted by a pseudo-separatrix through which the system state can possibly pass and, in so doing, escape from the potential well [17]. In fact, this is the only mechanism for escape from the potential well and the phase space flux, defined as the quantity of phase space transported across the pseudo-separatrix, is a measure of the possibility of and the time required for escape from the well. Mean rates of escape for potential wells are treated from this viewpoint in [15].

Phase space transport is richly interrelated with Melnikov processes, chaotic dynamics and system stability. First, the time average of the positive part of the Melnikov process or, equivalently, the corresponding ensemble average provides a leading-order description of the phase space flux across the pseudo-separatrix [2,6,17]. This quantity, called the flux factor, is, moreover, readily computed for a wide range of system potentials and forcing models. Second, Melnikov functions and processes are closely related to chaotic dynamics [2,16,17]: for a given realization of the forcing process, chaos occurs only if the corresponding realization of the Melnikov process has zeros. The phase space flux in turn has been interpreted as the system's "propensity" for chaos [2]. Finally, the ratio of fluxes from competing potential wells has been identified as a measure of the relative stability of the wells [8]. In this sense, the stability of a well is related, through the phase space flux across the well's pseudo-separatrix, to the system tendency to escape from the well.

Because the system state must be transported across the pseudo-separatrix in order to escape from the potential well, controlling the flux is a means to delay or promote escape and, equivalently, to stabilize or destabilize the potential well. The flux factor is proposed here as an objective function for control of a weakly forced system against escape from a potential well. Weak perturbations constituted of stationary forcing and linear damping are treated with special atttention given to colored Gaussian forcing. We initially consider multiplicative closed-loop control and show that, with the flux factor as the objective, such controls have equivalent open-loop counterparts in which the control process evolves in time independently of the system state.

The remainder of the paper is presented in five sections. Section 2 introduces the system, control and forcing models to be treated. Also in this section, we establish the equivalence of closed-loop multiplicative and open-loop additive controls. In Section 3, we obtain a necessary and sufficient condition for open-loop control to increase the stability of the system state within a potential well. Section 4 contains an equivalent condition in terms of the control lag, identifying the relationship between the optimal control strength and the lag. Section 5 illustrates the theory with a numerical example based on the Duffing oscillator. Extensions of the theory are briefly indicated in Section 6.

## 2. Model

We consider the one-dimensional Newtonian system $\ddot{x} = -V'(x)$ where $V(x)$ is a multistable potential with a hyperbolic fixed point and a homoclinic orbit $z_s(t) = (x_s(t), \dot{x}_s(t))$ [12]. The homoclinic orbit $z_s(t)$ is the separatrix for a corresponding potential well of $V(x)$. If the system $\ddot{x} = -V'(x)$ is modified to include a time-dependent perturbative term, and if this term is sufficiently weak, the separatrix becomes "porous" as noted above, forming a proximate pseudo-separatrix [16, p.528]. For suitable values of the system parameters the system state can penetrate the pseudo-separatrix and hence escape the potential well. Thus we consider the weakly ($0 < \varepsilon \ll 1$) perturbed system

$$\ddot{x} - V'(x) + \varepsilon[\gamma F(t) - k\dot{x} - \alpha C(t)] \tag{1}$$

in which the stochastic process $F(t)$ is an externally applied force. The forcing $F(t)$ introduces the possibility of escape, reducing the stability of the potential well. The process $C(t)$ in (1) is a control process designed to counteract $F(t)$, delaying or eliminating escapes and increasing the stability of the system state within the well. The perturbed system considered here is subject to weak damping as represented by the term $-k\dot{x}$ in (1). Intuitively, the damping represents passive resistance internal to the system against escape in contradistinction to the control's active external resistance. For simplicity only linear damping is considered. The parameters $\gamma, k, \alpha \geq 0$ in (1) fix the relative amounts of forcing, damping and control in the system.

The forcing $F(t)$ in (1) is assumed to be a wide-sense stationary process with zero mean and autocovariance $c_F(t) = \mathrm{Cov}[F(s), F(s+t)]$ [18]. Its spectral density

$$\hat{c}_F(\nu) = \int_{-\infty}^{\infty} c_F(t)\mathrm{e}^{-\mathrm{j}\nu t}\,\mathrm{d}t$$

(the Fourier transform of $c_F(t)$) is assumed to exist so that $F(t)$ can be written $F(t) = \mathcal{F}[\dot{W}](t) = (f * \dot{W})(t)$ where $\mathcal{F}$ is a time-invariant linear filter with square-integrable impulse response $f(t)$, $\dot{W}(t)$ is a white noise process [18, p.112] and $(f * \dot{W})(t)$ is the convolution of $f(t)$ and $\dot{W}(t)$. The process $F(t) = \mathcal{F}[\dot{W}](t)$ can be expressed as the stochastic integral

$$F(t) = \int_{-\infty}^{\infty} f(t-s)\,\mathrm{d}W(s), \tag{2}$$

where $W(t)$ is a process with orthogonal increments satisfying

$$E[(W(t_1) - W(t_2))(W(t_3) - W(t_4))] = l([t_1, t_2] \cap [t_3, t_4])$$

for all $t_1 \leq t_2$ and $t_3 \leq t_4$. Here $l([s,t]) = t - s$ denotes the length of the interval $[s,t]$. Model (2) includes Gaussian, shot noise and dichotomous noise forcing models and is, for example, a colored Gaussian process if $W(t)$ is Brownian motion. Integrals of the type (2) have the property that

$$\mathrm{Cov}\left[\int_{-\infty}^{\infty} w_1(t)\,\mathrm{d}W(t), \int_{-\infty}^{\infty} w_2(t)\,\mathrm{d}W(t)\right] = \int_{-\infty}^{\infty} w_1(t)w_2(t)\,\mathrm{d}t \tag{3}$$

provided $w_1(t)$ and $w_2(t)$ are square-integrable [4]. This last condition will be met in our uses of (3) since $f(t)$ is square-integrable. Given (2), the autocovariance of $F(t)$ is

$$c_F(t) = \int_{-\infty}^{\infty} f(s)f(s+t)\,\mathrm{d}s$$

with $\hat{c}_F(\nu) = |\hat{f}(\nu)|^2$. Here and throughout we adhere to the convention that, for example, $\hat{f}(\nu)$ is the Fourier transform of $f(t)$.

If the control and forcing processes $C(t)$ and $F(t)$ are uncorrelated then the presence of $C(t)$ in (1) cannot decrease the rate of escape from the potential well. On the other hand, if the path realized by $F(t)$ is completely known so that $C(t)$ can be made perfectly correlated with $F(t)$, then $C(t)$ completely cancels the effect of $F(t)$ for $\alpha = \gamma$. Neither situation is realistic; the control $C(t)$ is, in the first instance, ineffectual and, in the second, impractical. Reasonably, the control $C(t)$ should be able at least to track low frequency components of $F(t)$, perhaps with some lag $\ell > 0$. We therefore model the control process by

$$C(t) = \mathcal{G}[F](t - \ell), \tag{4}$$

where $\ell \geq 0$ is a constant representing the control lag and $\mathcal{G}$ is a causal, time-invariant, linear filter with impulse response $g(t)$ and transfer function $\hat{g}(\nu)$. Typically, the control filter $\mathcal{G}$ will be low-pass, reflecting the inability of $C(t)$ to track high frequency components of $F(t)$. It follows from (4) that

$$C(t) = (g * \delta_\ell * F)(t) = (g * \delta_\ell * f * \dot{W})(t),$$

where $\delta(t)$ is the Dirac-$\delta$ function and $\delta_\ell(t) = \delta(t - \ell)$. The control process $C(t)$ can therefore be expressed as the integral

$$C(t) = \int_{-\infty}^{\infty} (g * \delta_\ell * f)(t - s)\, \mathrm{d}W(s).$$

Three classes of excitation can be identified for system (1) with increasing degrees of generality. An excitation $\chi X(t)$ such as $\gamma F(t)$ or $\alpha C(t)$ in (1) is called additive if $\chi$ is a constant independent of the system state. More generally, if $\chi = \chi(x, \dot{x})$ varies with system state, then the excitation $\chi(x, \dot{x}) X(t)$ is multiplicative. Still more general are excitations of the form $w(X(t), x, \dot{x})$. The effect of an external excitation on the transport of phase space and, more specifically, on the flux factor is mediated by a filter $\Theta$ called the orbit filter because of its dependence on the geometry of the homoclinic orbit [6,7]. For the excitation $w(X(t), x, \dot{x})$ this filter takes the form [10, p.85]

$$\Theta[w](t) = \int_{-\infty}^{\infty} \dot{x}_s(\tau) w(X(\tau + t), x_s(\tau), \dot{x}_s(\tau))\, \mathrm{d}\tau.$$

The filter $\Theta$ is here neither linear nor time-invariant. For multiplicative excitations, though, including the additive case, $\Theta$ is both linear and time-invariant,

$$\Theta[\chi X](t) = \int_{-\infty}^{\infty} \dot{x}_s(\tau) \chi(x_s(\tau), \dot{x}_s(\tau)) X(\tau + t)\, \mathrm{d}\tau.$$

In fact, in this case the orbit filter response can be written as the convolution integral:

$$\Theta[\chi X](t) = \int_{-\infty}^{\infty} h(t - \tau) X(\tau)\, \mathrm{d}\tau,$$

where

$$h(t) = \chi(x_s(-t), \dot{x}_s(-t)) \dot{x}_s(-t) \tag{5}$$

is the orbit filter impulse response [7]. Our analysis, being based on spectral properties of the excitation and the orbit filter, requires linearity and time-invariance and is therefore restricted to multiplicative excitations. A schematic

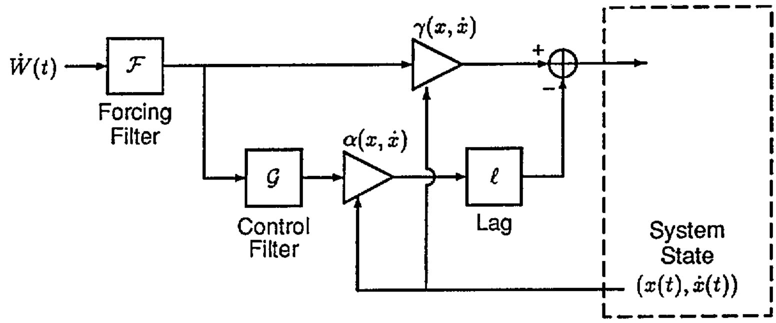


Fig. 1. Dynamical system with multiplicative forcing and control.

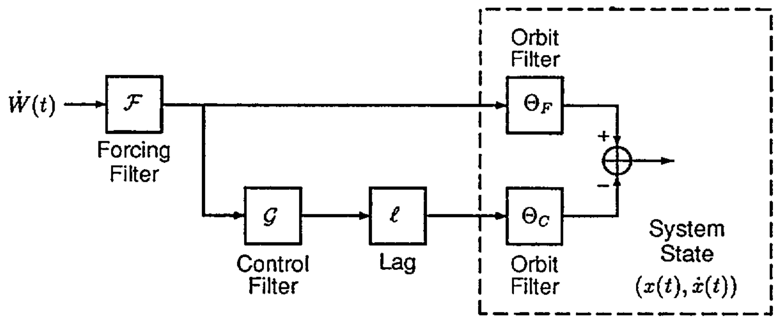


Fig. 2. Equivalent dynamical system with multiplicative forcing and control given flux factor as the control objective.

diagram of system (1) with multiplicative forcing and control is shown in Fig. 1. Multiplicative controls depend on the current system state $(x, \dot{x})$ and are a form of closed-loop control. If the control is additive then there is no dependence on system state, there is no feedback path and the control is open-loop.

Whether additive or multiplicative, the effect of the excitation $\chi X(t)$ on the flux factor is determined by the time-invariant linear orbit filter $\Theta$. Thus *if the flux factor is the control objective*, the multiplicatively forced and controlled system in Fig. 1 can be represented as in Fig. 2. For system (1) the impulse response $h(t)$ of the orbit filter $\Theta$ associated with the multiplicative excitation $\chi(x, \dot{x})X(t)$ is as in (5). Thus, as shown in Fig. 2, the excitations representing forcing and control have associated with them orbit filters $\Theta_{\mathrm{F}}$ and $\Theta_{\mathrm{C}}$ with different impluse responses. If the excitation $\chi X(t)$ is additive, then the impulse reponse in (5) is $h(t) = \chi \dot{x}_s(-t)$. So for additive excitations $\gamma F(t)$ and $\alpha C(t)$ the difference in $\Theta_{\mathrm{F}}$ and $\Theta_{\mathrm{C}}$ is one only of multiplicative constants and it is convenient to speak of a common orbit filter $\Theta$ with impulse response $h(t) = \dot{x}_s(-t)$. A diagram of the additively forced and controlled system is shown in Fig. 3. Observe that for the multiplicatively forced and controlled system in Fig. 2, if we choose $\Theta = \Theta_{\mathrm{F}}$ and $\mathcal{G}'$ exists such that $\mathcal{G}\Theta_{\mathrm{F}} = \mathcal{G}\Theta_{\mathrm{C}}$ then the system can be represented as shown in Fig. 4. Remarkably, the representation in Fig. 4 is then the same as that in Fig. 3 for the additively forced and controlled system. This shows that if the flux factor is the control objective and the system is a weakly perturbed Newtonian system as in (1), then multiplicative control and/or forcing has an additive representation and means, in particular, that multiplicative closed-loop control holds no advantage over open-loop control.

The filter $\mathcal{G}'$ satisfying $\mathcal{G}'\Theta_{\mathrm{F}} = \mathcal{G}\Theta_{\mathrm{C}}$ does not exist for all combinations of orbit filters $\Theta_{\mathrm{F}}$ and $\Theta_{\mathrm{C}}$ and control filters $\mathcal{G}$. While our results are not substantively changed in the absence of $\mathcal{G}'$, their derivation is significantly complicated by the need to account for two orbit filters – $\Theta_{\mathrm{F}}$ and $\Theta_{\mathrm{C}}$ – rather than just $\Theta$. For example, the vector process $(\Theta[F](t), \Theta[C](t))$ associated with (6) below becomes $(\Theta_{\mathrm{F}}[F](t), \Theta_{\mathrm{C}}[C](t))$. For simplicity, then, we assume $\mathcal{G}'$ to exist and restrict our attention to the open-loop control problem represented in Fig. 3.

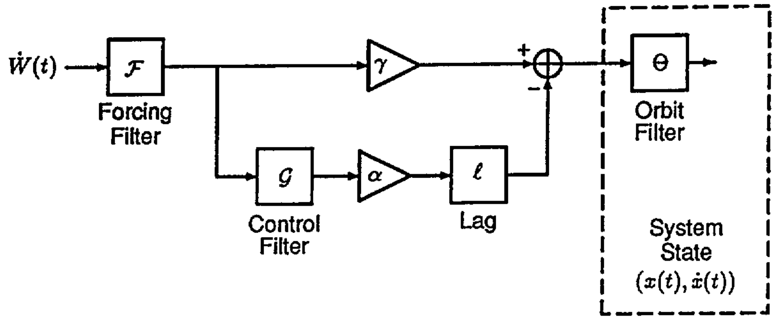


Fig. 3. Dynamical system with additive forcing and control given flux factor as the control objective.

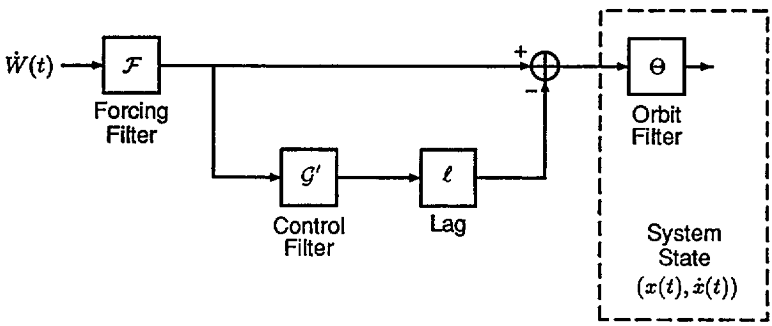


Fig. 4. Equivalent open-loop (additive) representation of the multiplicatively forced and controlled dynamical system given flux factor as the control objective.

Three filters $\mathcal{F}$, $\mathcal{G}$ and $\Theta$ with corresponding impulse responses $f(t)$, $g(t)$ and $h(t)$ have been introduced. The filter $\mathcal{F}$ models the spectrum of the forcing $F(t)$ and need not be causal. The filter $\Theta$ which we associate with the homoclinic orbit $z_s(t)$ is necessarily noncausal because its impulse response $h(t) = \dot{x}_s(-t)$ is nonzero for $t < 0$. Neither case presents a logical problem as neither filter is itself directly identified with any physical process or structure. The control filter $\mathcal{G}$, however, represents some mechanical or electronic device. It must therefore be causal with real impulse response function $g(t) = 0$ for $t < 0$.

## 3. Condition for increased stability

The phase space flux associated with the homoclinic orbit $z_s(t)$ of system (1) can be expressed as $\varepsilon\Xi + \mathrm{O}(\varepsilon^2)$ [13] where the flux factor $\Xi$ is [6]

$$\Xi = E[(\gamma X_1 - \alpha X_2 - kA)^+] \tag{6}$$

with $x^+$ denoting the positive part of the real number $x$; $x^+ = x$ for $x > 0$ and $x^+ = 0$ otherwise. The joint distribution of the random variables $X_1$ and $X_2$ in (6) is the same as the marginal distribution of the vector process $(\Theta[F](t), \Theta[C](t))$ and $A$ is the integral

$$A = \int_{-\infty}^{\infty} \dot{x}_s^2(t)\,\mathrm{d}t \tag{7}$$

$$= 2 \int_0^{x_m} \dot{x}_s \, dx_s, \tag{8}$$

where $x_m$ is the maximum of $x_s(t)$. Expression (6) depends on certain technical properties – ergodicity and uniform continuity, in particular – of the forcing beyond those related to wide-sense stationarity given above. These issues are thoroughly treated in [6,7]. Integral (8) identifies $A$ as the area of the interior of the homoclinic orbit $z_s(t)$. For small $\varepsilon$ the flux factor $\Xi$ in (6) is proportional to the time-averaged amount of phase space transported across the pseudo-separatrix [17]. The flux factor is as discussed above a measure of phase space transport across the pseudo-separatrix and hence of escape.

The processes $F(t)$ and $C(t) = \mathcal{G}[F](t - \ell)$ are jointly wide-sense stationary and the orbit filter $\Theta$ is linear and time-invariant so $X_1$ and $X_2$ in (6) are zero mean random variables with covariance matrix

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}. \tag{9}$$

The control $C(t)$ to some degree cancels $F(t)$ only if $C(t)$ and $F(t)$ are positively correlated; if negatively correlated, $C(t)$ is useless. We therefore ignore negatively correlated controls and, with no loss of generality, assume $\sigma_{12} > 0$ in (9). The random variable $\gamma X_1 - \alpha X_2$ is zero mean with variance

$$\sigma^2 = \mathrm{Var}[\gamma X_1 - \alpha X_2] = \gamma^2\sigma_1^2 + \alpha^2\sigma_2^2 - 2\gamma\alpha\sigma_{12}. \tag{10}$$

The flux factor $\Xi$ can then be written

$$\Xi = E[(\sigma Z - kA)^+], \tag{11}$$

where $Z = (\gamma X_1 - \alpha X_2)/\sigma$. Letting $F_Z(z) = P(Z \le z)$ be the cumulative distribution function (c.d.f.) of $Z$ we have further that

$$\Xi = kA\eta(\sigma/kA), \tag{12}$$

where by an elementary calculation

$$\eta(x) = x \int_{1/x}^{\infty} [1 - F_Z(z)] \, dz \tag{13}$$

for $x > 0$. This function has the following easily established properties: $\eta(x)$ is convex with $\eta(0) = \eta'(0) = 0$, $\eta'(x) \ge 0$, $\eta''(x) \ge 0$ and

$$\eta'_\infty \equiv \lim_{x\to\infty} \eta'(x) = \int_0^{\infty} [1 - F_Z(z)] \, dz.$$

Furthermore, $\eta(x)$ has the piece-wise linear approximation

$$\eta(x) \doteq \begin{cases} 0, & 0 \le x \le x_0, \\ \eta'_\infty(x - x_0), & x > x_0, \end{cases} \tag{14}$$

where $x_0 = (1 - F_Z(0))/\eta'_\infty$. The flux factor $\Xi$ is a nondecreasing function of $\sigma$ since $\eta'(x) \ge 0$. Therefore the control $C(t)$ reduces the flux factor $\Xi$ in (12) and increases the stability of the system state within the potential well only if the introduction of $C(t)$ reduces $\sigma^2$ in (10) below its value $\gamma^2\sigma_1^2$ for $\alpha = 0$.

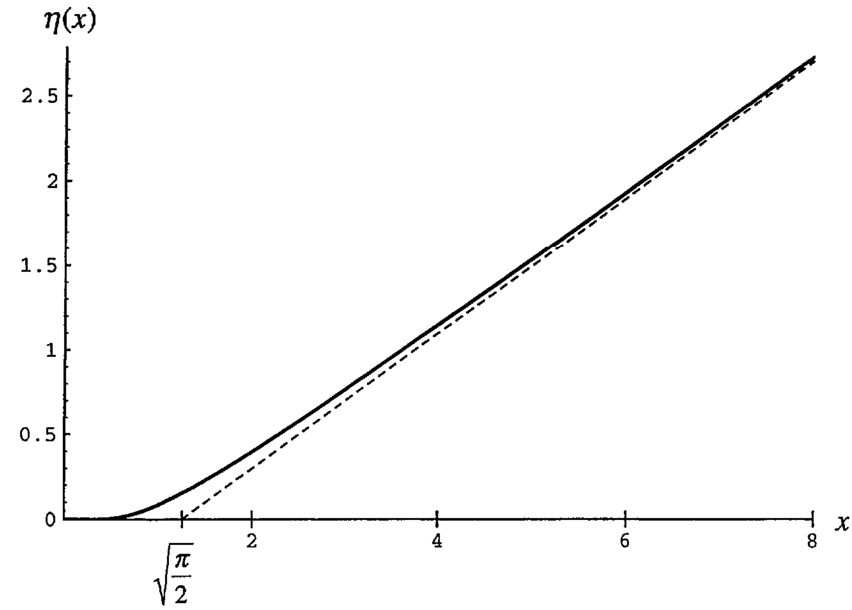


Fig. 5. A plot of $\eta(x) = x\phi(1/x) + \Phi(1/x) - 1$ and its linear asymptotic approximation.

In the special case of Gaussian forcing $F(t)$, the control $C(t)$ is also Gaussian and $X_1$ and $X_2$ in (6) are bivariate Gaussian random variables. Then $Z$ in (11) is a standard Gaussian random variable and $\eta(x)$ in (13) is [6]

$$\eta(x) = x\phi(1/x) + \Phi(1/x) - 1, \tag{15}$$

where $\phi(z) = (2\pi)^{-1/2}\exp(-\frac{1}{2}z^2)$ is the standard Gaussian density and $\Phi(z)$ is the corresponding c.d.f. The function $\eta(x)$ in (15) is shown in Fig. 5 together with its asymptotic approximation

$$\eta(x) \doteq \frac{x}{\sqrt{2\pi}} - \frac{1}{2}, \quad x > \sqrt{\frac{\pi}{2}}$$

derived from (14). The derivative of $\eta(x)$ in (15) is $\eta'(x) = \phi(1/x) > 0$ for all $x > 0$ hence, as seen in Fig. 5, $\eta(x)$ is a strictly increasing function of $x$. Thus in this case, as in all cases for which the c.d.f. $F_Z(z)$ of $Z$ is strictly increasing for all $-\infty < z < \infty$, the control $C(t)$ reduces the flux factor $\Xi$ if and only if it reduces $\sigma^2$. In other words, if the support of $Z$ is $(-\infty, \infty)$ then every decrease in $\sigma^2$ produces a corresponding decrease in the flux factor $\Xi$; if $Z$ has bounded support, then decreases in $\sigma^2$ past a certain point produce no further decrease in $\Xi$.

Returning to the general case of wide-sense stationary forcing, we define

$$\Delta = \sigma^2 - \gamma^2\sigma_1^2 = \alpha^2\sigma_2^2 - 2\alpha\gamma\sigma_{12}.$$

The control $C(t)$ increases the stability of the system state within the potential well only if $\Delta < 0$. We have, in turn, that $\Delta < 0$ if and only if

$$\frac{\alpha}{\gamma} < 2\frac{\sigma_{12}}{\sigma_2^2}. \tag{16}$$

Condition (16) on the relative control strength $\alpha/\gamma$ is necessary for the control to increase stability. The optimal relative control strength satisfying (16) is found by elementary calculus to be

$$\left.\frac{\alpha}{\gamma}\right|_{\text{optimal}} = \frac{\sigma_{12}}{\sigma_2^2} \tag{17}$$

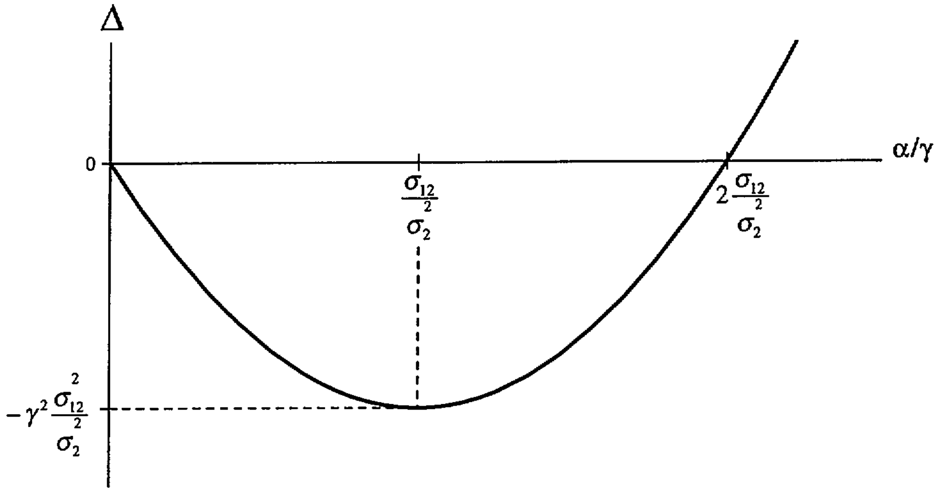


Fig. 6. Change $\Delta$ in $\sigma^2$ as a function of the relative control signal strength $\alpha/\gamma$.

in which case $\Delta = -\gamma^2\sigma_{12}^2/\sigma_2^2$. See Fig. 6. The reduction in the flux factor achieved at the optimal relative control strength (17) is

$$kA\eta\left(\frac{\gamma\sigma_1}{kA}\right) - kA\eta\left(Q\frac{\gamma\sigma_1}{kA}\right), \tag{18}$$

where

$$Q = \sqrt{1 - \sigma_{12}^2/\sigma_1^2\sigma_2^2}.$$

By the Cauchy–Schwartz inequality $\sigma_{12}^2 < \sigma_1^2\sigma_2^2$ so $0 \le Q \le 1$. For $Q\gamma\sigma_1/(kA) > x_0$ (this condition derives from approximation (14)) the maximum reduction (18) in the flux factor is, from (14), approximately

$$kA\eta\left(\frac{\gamma\sigma_1}{kA}\right) - kA\eta\left(Q\frac{\gamma\sigma_1}{kA}\right) \doteq (1-Q)\gamma\sigma_1\eta_\infty. \tag{19}$$

This approximation is exact in the limiting case of no damping; that is, the maximum reduction in the flux factor is $(1-Q)\gamma\sigma_1\eta_\infty$ for $k = 0$. Approximation (19) suggests that $1 - Q$ be treated as a measure of the proportional reduction achievable for the flux factor by a given control. The absence of the damping constant $k$ from $1 - Q$ implies that while $k$ does play a role in the flux factor, it is secondary to that played by the second order statistics of the forcing and control processes. This is unsurprising since the damping resists the action of both $F(t)$ *and* $C(t)$.

## 4. Role of control lag

We begin by expressing the components $\sigma_1^2$, $\sigma_2^2$ and $\sigma_{12}$ of (9) as integrals. Let $c_{\Theta[F]}(t) = \mathrm{Cov}[\Theta[F](\tau + t), \Theta[F](\tau)]$ be the autocovariance of the filtered process $\Theta[F](t)$. The spectral density of $\Theta[F](t)$ is the Fourier transform

$$\hat{c}_{\Theta[F]}(\nu) = |\hat{h}(\nu)|^2|\hat{f}(\nu)|^2$$

of $c_{\Theta[F]}(t)$ and

$$c_{\Theta[F]}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{c}_{\Theta[F]}(\nu)\mathrm{e}^{\mathrm{j}\nu t}\,\mathrm{d}\nu = \frac{1}{\pi}\int_{0}^{\infty} |\hat{h}(\nu)|^2|\hat{f}(\nu)|^2\cos\nu t\,\mathrm{d}\nu.$$

In particular,

$$c_{\Theta[F]}(0) = \frac{1}{\pi}\int_{0}^{\infty} |\hat{h}(\nu)|^2|\hat{f}(\nu)|^2\,\mathrm{d}\nu.$$

Therefore,

$$\sigma_1^2 = \mathrm{Var}[X_1] = \mathrm{Var}[\Theta[F](t)] = c_{\Theta[F]}(0) = \frac{1}{\pi}\int_{0}^{\infty} |\hat{h}(\nu)|^2|\hat{f}(\nu)|^2\,\mathrm{d}\nu \equiv \frac{J_0}{\pi}. \tag{20}$$

For the case in which $F(t) = \dot{W}(t)$ is a white noise process, (20) admits a significant simplification. Using (3) and (7),

$$\sigma_1^2 = \mathrm{Var}[\Theta[\dot{W}(t)]\,\mathrm{d}\nu = \int_{-\infty}^{\infty} h^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \dot{x}_s^2(t)\,\mathrm{d}t = A,$$

where $A$ is the area of the region in phase space enclosed by the homoclinic orbit $z_s(t) = (x_s(t), \dot{x}_s(t))$.

Let $c_{\Theta[\mathcal{G}[F]]}(t)$ and $\hat{c}_{\Theta[\mathcal{G}[F]]}(\nu)$ be the autocovariance and spectral density of the process $\Theta[\mathcal{G}[F]](t)$ which results from filtering $F(t)$ by $\mathcal{G}$ and $\Theta$ successively. It follows that $\hat{c}_{\Theta[\mathcal{G}[F]]}(\nu) = |\hat{h}(\nu)|^2|\hat{g}(\nu)|^2|\hat{f}(\nu)|^2$. Therefore,

$$\begin{aligned}\sigma_2^2 &= \mathrm{Var}[X_2] = \mathrm{Var}[\Theta[\mathcal{G}[F]](t-\ell)] = \mathrm{Var}[\Theta[\mathcal{G}[F]](t)] = c_{\Theta[\mathcal{G}[F]]}(0)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{h}(\nu)|^2|\hat{g}(\nu)|^2|\hat{f}(\nu)|^2\,\mathrm{d}\nu = \frac{1}{\pi}\int_{0}^{\infty} |\hat{h}(\nu)|^2|\hat{g}(\nu)|^2|\hat{f}(\nu)|^2\,\mathrm{d}\nu \equiv \frac{J_1}{\pi}.\end{aligned} \tag{21}$$

Using (3) followed by Parseval's theorem [11],

$$\begin{aligned}\sigma_{12} &= \mathrm{Cov}[X_1, X_2] = \mathrm{Cov}[\Theta[F](t), \Theta[\mathcal{G}[F]](t-\ell)]\\ &= \int_{-\infty}^{\infty} (h * f)(s)(h * g * f * \delta_\ell)(s)\,\mathrm{d}s\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(-\nu)\hat{f}(-\nu)\hat{h}(\nu)\hat{g}(\nu)\hat{f}(\nu)\mathrm{e}^{-\mathrm{j}\nu\ell}\,\mathrm{d}\nu\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{h}(\nu)|^2|\hat{f}(\nu)|^2\hat{g}(\nu)\mathrm{e}^{-\mathrm{j}\ell\nu}\,\mathrm{d}\nu.\end{aligned} \tag{22}$$

Write $\hat{g}(\nu) = \mathcal{R}(\nu) + \mathrm{j}\mathcal{I}(\nu)$ where $\mathcal{R}(\nu)$ and $\mathcal{I}(\nu)$ are the real and imaginary parts of $\hat{g}(\nu)$. The impulse response $g(t)$ of the control filter $\mathcal{G}$ is real so the functions $\mathcal{R}(\nu)$ and $\mathcal{I}(\nu)$ are even and odd, respectively. Then

$$
\begin{aligned}
\sigma_{12} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{h}(\nu)|^2 |\hat{f}(\nu)|^2 [\mathcal{R}(\nu) + \mathrm{j}\mathcal{I}(\nu)][\cos \ell\nu - \mathrm{j} \sin \ell\nu] \, \mathrm{d}\nu \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{h}(\nu)|^2 |\hat{f}(\nu)|^2 [\mathcal{R}(\nu) \cos \ell\nu + \mathcal{I}(\nu) \sin \ell\nu] \, \mathrm{d}\nu \\
&= \frac{1}{\pi} \int_{0}^{\infty} |\hat{h}(\nu)|^2 |\hat{f}(\nu)|^2 \mathcal{R}(\nu) \cos \ell\nu \, \mathrm{d}\nu + \frac{1}{\pi} \int_{0}^{\infty} |\hat{h}(\nu)|^2 |\hat{f}(\nu)|^2 \mathcal{I}(\nu) \sin \ell\nu \, \mathrm{d}\nu.
\end{aligned} \tag{23}
$$

We have $\cos x = 1 + \mathrm{o}(x)$ and $\sin x = x + \mathrm{o}(x^2)$ in the limit as $x \to 0$ so

$$
\begin{aligned}
\sigma_{12} &= \frac{1}{\pi} \int_{0}^{\infty} |\hat{h}(\nu)|^2 |\hat{f}(\nu)|^2 \mathcal{R}(\nu) \, \mathrm{d}\nu + \frac{\ell}{\pi} \int_{0}^{\infty} |\hat{h}(\nu)|^2 |\hat{f}(\nu)|^2 \mathcal{I}(\nu) \nu \, \mathrm{d}\nu + \mathrm{o}(\ell) \\
&\equiv \frac{J_2}{\pi} - \frac{J_3}{\pi}\ell + \mathrm{o}(\ell) \doteq \frac{J_2}{\pi} - \frac{J_3}{\pi}\ell
\end{aligned} \tag{24}
$$

for small lags $\ell$. In cases of interest $J_2 > 0$ and $J_3 > 0$. Using approximation (24) for $\sigma_{12}$ we have

$$
\sigma^2 = \gamma^2\sigma_1^2 + \alpha^2\sigma_2^2 - 2\gamma\alpha\sigma_{12} \doteq \gamma^2 \frac{J_0}{\pi} + \alpha^2 \frac{J_1}{\pi} - 2\gamma\alpha\frac{J_2}{\pi} + 2\gamma\alpha\ell\frac{J_3}{\pi}. \tag{25}
$$

If the lag $\ell$ is small, condition (16) for increased stability is approximately

$$
\frac{J_1}{2}\frac{\alpha}{\gamma} + J_3\ell < J_2. \tag{26}
$$

Approximation (24) from which condition (26) follows was made for convenience. While higher-order terms are readily included in (24), they do not substantively change the overall result.

Condition (26) is shown in Fig. 7. This condition imposes upper bounds on both of the fundamental control parameters: the relative strength $\alpha/\gamma$ of the control signal and the lag $\ell$. Naturally, to delay and minimize escapes, the lag $\ell$ should be small since the smaller the lag in $C(t)$ the better it can cancel the forcing $F(t)$. The upper bound on $\ell$ indicates that large lags are, as expected, useless and justifies using a small-lag approximation. The upper bound on the relative strength $\alpha/\gamma$ of the control signal is less intutive. It might be expected that the stronger the control signal the more of the forcing it could cancel. However, the control $C(t)$ in system (1) actually has two competing effects. The first, represented by the term $-2\gamma\alpha J_2/\pi$ in (25), has the effect of decreasing $\sigma^2$ and contributes to increased stability through cancellation of $F(t)$. The second effect of $C(t)$ is represented by the terms $\alpha^2 J_1/\pi + 2\gamma\alpha\ell J_3/\pi$ in (25). For cases of interest, these terms are positive and increase $\sigma^2$. Their presence in (25) is explained by recognizing that $C(t)$ is an additional external forcing to the system and, to the extent that it is mismatched to $F(t)$ because of lag or spectral difference, will promote instability and escape.

Using approximation (24), the optimal relative control strength is

$$
\left.\frac{\alpha}{\gamma}\right|_{\text{optimal}} = \frac{J_2}{J_1} - \ell\frac{J_3}{J_1}. \tag{27}
$$

This is the straight line which bisects the shaded region in Fig. 7. The degree to which stability is increased is, for the approximate optimal relative control strength (27), the change (18) in the flux factor with

$$
Q = \sqrt{1 - \frac{(J_2 - \ell J_3)^2}{J_0 J_1}}. \tag{28}
$$

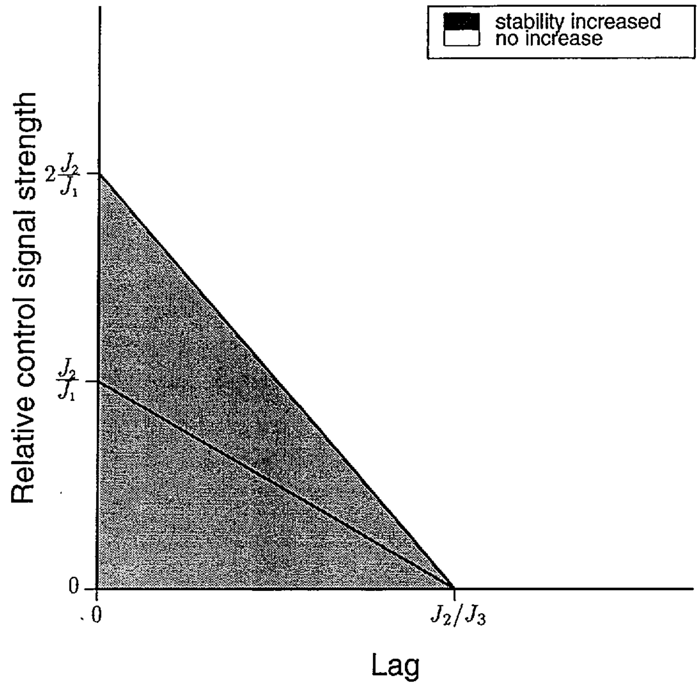


Fig. 7. Combinations of relative control signal strength $\alpha/\gamma$ lag $\ell$ within the shaded region reduce phase space transport and increase stability of the system state within the potential well. Outside the shaded region stability is not increased. The solid line in the interior of the shaded region identifies the optimal choice of relative strength $\alpha/\gamma$ for a given lag $\ell$.

For a given relative control strength $\alpha/\gamma$ and lag $\ell$, the effectiveness of the control $\alpha C(t)$ depends on the control filter $\mathcal{G}$ and, in particular, on the control filter's gain $|\hat{g}(\nu)| = (\mathcal{R}^2(\nu) + \mathcal{I}^2(\nu))^{1/2}$ and its phase $\tan^{-1}(\mathcal{I}(\nu)/\mathcal{R}(\nu))$ which governs intrinsic lag of the filter. A useful control is one for which the average control power is small relative to the average forcing power. This implies a control filter with spectral power only at the frequencies needed to counteract the filtered forcing $\Theta[F](t)$ remaining after filtering by the orbit filter $\Theta$. This, in addition to minimizing $Q$ in (28), guides the search for a suitable control filter. An example is given to illustrate how a simple "off-the-shelf" parametric filter might be optimized for a given control problem.

## 5. Numerical example

We consider a Duffing oscillator described by Eq. (1) with the double-well potential $V(x) = x^4/4 - x^2/2$. This system is bistable with a hyperbolic fixed point $z_0$ at $(x, \dot{x}) = (0, 0)$ in phase space. Associated with the wells of $V(x)$ are two orbits – left and right – homoclinic to $z_0$. The velocity component of the right homoclinic orbit is $\dot{x}_s(t) = -\sqrt{2}\,\mathrm{sech}\, x \tanh x$. It follows that $A = \frac{4}{3}$, the orbit filter impulse response is $h(t) = \dot{x}_s(-t) = \sqrt{2}\,\mathrm{sech}\, x \tanh x$, and the modulus, or gain, of the orbit filter transfer function is [6]

$$|\hat{h}(\nu)| = \sqrt{2}\pi\nu \ \mathrm{sech}\ \tfrac{1}{2}\pi\nu.$$

The spectral density of the forcing in this example is broadband with $\hat{c}_F(\nu) = 1_{(-B,B)}(\nu)$ while the control filter $\mathcal{G}$ has the piecewise linear impulse response $g(t)$ shown in Fig. 8. The aim of this example is to choose the parameters

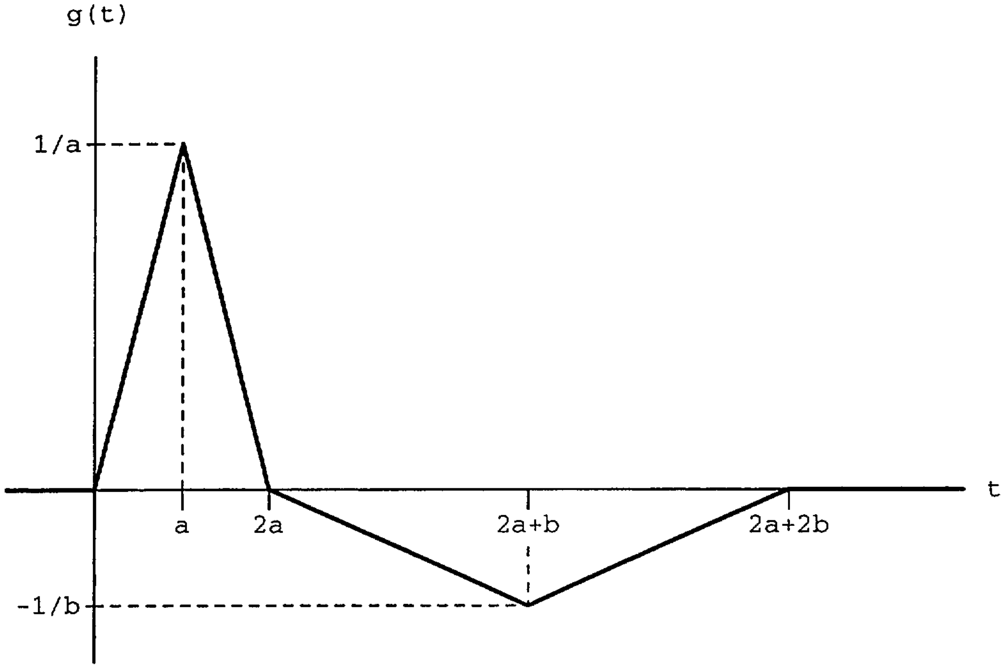


Fig. 8. Two-parameter example of a control filter impulse response with initial response and recoil.

$a$, $b$ of $g(t)$ to maximize $1-Q$ in (19) using expression (28). For this we need to know neither the damping constant $k$ nor anything of the forcing other than its spectral density.

The parameters of $g(t)$ determine the duration of the filter's intital response and recoil. In particular, for $a < b$, $a$ may be thought of as the intrinsic lag (additional to the explicitly imposed lag $\ell$) of the filter. This choice of filter is driven by several considerations. First, it is a realistic example of an easily implemented recursive digital filter. Second, the initial sharp pulse in its impulse response (the triangle centered at $t = a$ in Fig. 8) means the intrinsic lag will not be too large. Third, the two opposing pulses in its impulse response have equal area. Thus the zero-frequency gain of the control filter is matched to that of the orbit filter: $|\hat{g}(0)| = |\hat{h}(0)| = 0$. Finally, triangular pulses were chosen for the impulse response rather than, say, rectangular pulses so that the control's high frequency spectral content falls off rapidly. These last two points ensure that the spectral energy of the control will be concentrated primarily within the passband of the orbit filter and not wasted at near-zero or high frequencies. These spectral considerations, which lead to a more effective control with less wasted energy, are motivated by knowledge of the action of the orbit filter on $F(t)$ and $C(t)$.

The real and imaginary parts of the transfer function of $\mathcal{G}$ are

$$\mathcal{R}(\nu) = \mathrm{Sa}^2(\tfrac{1}{2}a\nu)\cos a\nu - \mathrm{Sa}^2(\tfrac{1}{2}b\nu)\cos(2a+b)\nu,$$
$$\mathcal{I}(\nu) = -\mathrm{Sa}^2(\tfrac{1}{2}a\nu)\sin a\nu + \mathrm{Sa}^2(\tfrac{1}{2}b\nu)\sin(2a+b)\nu,$$

where $\mathrm{Sa}(x) = \sin(x)/x$. The squared modulus of the transfer function $\hat{g}(\nu)$ is

$$\begin{aligned}|\hat{g}(\nu)|^2 &= \mathcal{R}^2(\nu) + \mathcal{I}^2(\nu)\\ &= \mathrm{Sa}^4(\tfrac{1}{2}a\nu) + \mathrm{Sa}^4(\tfrac{1}{2}b\nu) - 2\mathrm{Sa}^2(\tfrac{1}{2}a\nu)\mathrm{Sa}^2(\tfrac{1}{2}b\nu)\cos(a+b)\nu. \qquad (29)\end{aligned}$$

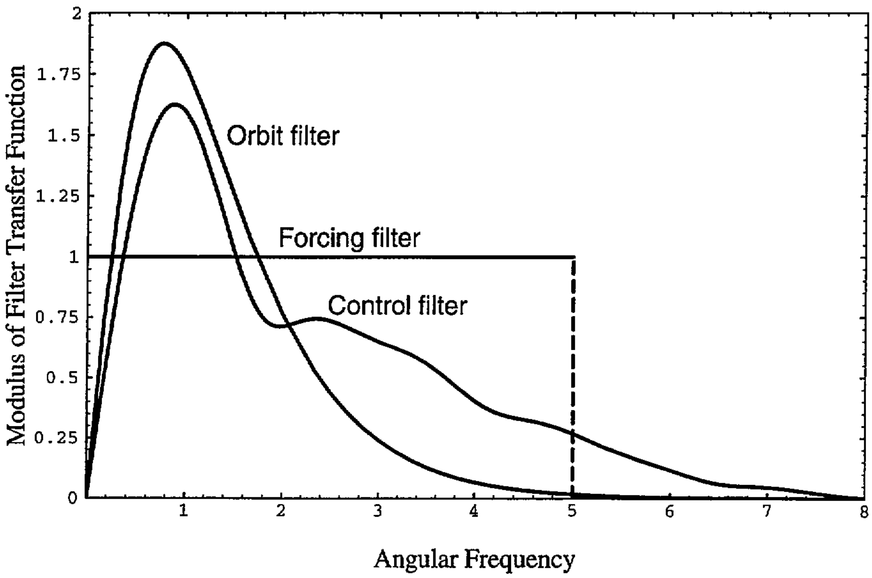


Fig. 9. Moduli of control ($a = 0.75$, $b = 2.25$), forcing and Duffing oscillator orbit filter transfer functions.

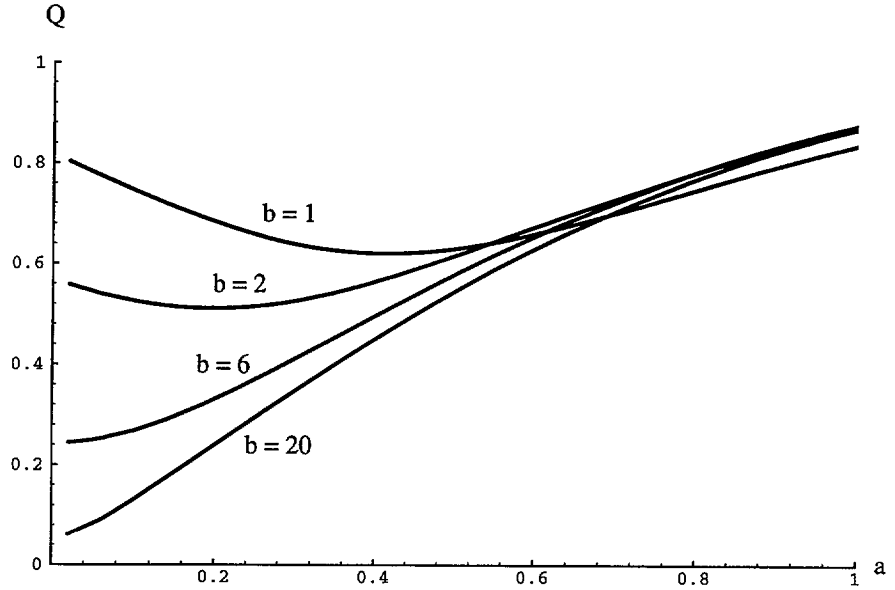


Fig. 10. $Q$ in the Duffing oscillator example for various values of control filter parameters $a$ and $b$.

The transfer function of the control filter $\mathcal{G}$ with $a = 0.75$ and $b = 2.25$ is displayed in Fig. 9 along with those of the forcing filter and the orbit filter. We calculate $J_0$, $J_1$ and $J_2$ for $B = 5$ and various values of $a$ and $b$. From these, $Q$ is calculated with the results shown in Fig. 10. These results assume that the control lag $\ell$ is zero; nonzero lags increase $Q$ in the fashion specified in (28). In (18) and, more evidently in (19), $1 - Q$ appears as an approximate fractional reduction in the flux factor. Hence, for $Q$ close to zero, control significantly reduces the flux and increases stability against escape. Fig. 10 shows that small $Q$ is possible in the present example for small $a$ and large $b$.

The average power of the control is an important practical consideration. In the present case, the average power of $C(t)$ is

$$\frac{1}{\pi}\int_0^\infty |\hat{g}(\nu)|^2 \hat{c}_F(\nu)\,d\nu = \frac{1}{\pi}\int_0^B |\hat{g}(\nu)|^2\,d\nu < \frac{1}{\pi}\int_0^\infty |\hat{g}(\nu)|^2\,d\nu \equiv \frac{K}{\pi}.$$

Thus the average control power as a fraction of the average power of the forcing $F(t)$ is bounded above by $K/B$. For a control filter with a given $K$, this is large or small depending on the bandwidth $B$ of the forcing spectrum.

The present example indicates the circumstances under which the proposed control scheme is anticipated to be effective and is analogous, for example, to successful applications of control in the field of active sonic noise reduction. In noise reduction, control is effective when active and passive techniques are efficiently combined [5]. Passive sound absorptive materials may be used, for example, to suppress high frequency sound components while active cancellation is employed against low frequency components. This efficient solution plays to the strengths of both methods: high frequency sound absorption requires little absorptive material relative to that needed at low frequencies while active cancellation is most effective at low frequencies where tracking is less of a problem. In the context of the present, more abstract example, the control $C(t)$ successfully counters much of the low frequency spectral content of $F(t)$. At high frequencies where $C(t)$ is less effective, the passive filtering inherent in the system orbit filter $\Theta$ dampens the high frequency components of $F(t)$. This suggests a strategy in which the system is designed with a low-pass orbit filter $\Theta$ chosen to reject as much of the forcing $F(t)$ as possible and in which the control $C(t)$ is designed to counteract the residual forcing $F(t)$ as it appears at the output of the orbit filter $\Theta$.

The conrol filter gain $|\hat{g}(\nu)|$, though important for effective control, reflects nothing of the filter's timing or phase. To see this in the example of the Duffing oscillator, note from (29) that $|\hat{g}(\nu)| \equiv |\hat{g}(\nu; a, b)|$ is symmetric in $a$ and $b$; $|\hat{g}(\nu; a, b)| = |\hat{g}(\nu; b, a)|$. Consider two cases: the first in which $a = 1/\lambda$ and $b = \lambda$ with $\lambda > 1$ and the second in which $a = \lambda$ and $b = 1/\lambda$. The control filter gain is the same in each case. But in the first case, the implicit filter lag is $a$ while in the second it is $2a + b$ – longer than in the first case. Moreover, in the first case the control $C(t)$ and the forcing $F(t)$ are positively correlated (the desired relationship) while in the second case they are negatively correlated. For both reasons – shorter lag and positive correlation – the parameter values in the first case are a much better choice than those of the second. This example illustrates the point that both timing and gain must be considered in the choice of control filter.

## 6. Final remarks

The control problem presented here can be extended in a variety of ways. A more realistic control signal might incorporate a noise term $n(t)$ independent of the forcing process as follows:

$$C(t) = \alpha\mathcal{G}[F](t - \ell) + n(t).$$

The above analysis easily adapts to this model. One finds that the noise $n(t)$ necessarily partially or completely cancels the increased stability provided by the control process. To a degree, this can be compensated for by increasing the control signal strength $\alpha$, reducing the lag $\ell$ or modifying the control filter $\mathcal{G}$. The linear damping represented by the term $-k\dot{x}$ in (1) can be replaced with other forms of damping with no substantive change in the analysis beyond modification of the constant $A$ in (6) and subsequent expressions. The control problem for systems with multiple hyperbolic fixed points connected by a heteroclinic cycle is very similar to that presented here and admits similar results. Finally, the theory worked out here applies generally to wide-sense stationary forcing models including as we have seen colored Gaussian forcing but including also, for example, dichotomous forcing and shot noise.

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